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Knödel walks in a Böhm-Hornik environment

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Abstract

Ideas of Knödel and Böhm-Hornik about walks in certain graphs, resembling the classical symmetric random walk on the integers, are combined. All the relevant generating functions (although occasionally quite involved) are made fully explicit. The treatment has an educational flavour as well.

Keywords: random walk, weighted edge, online bin-packing, generating functions, kernel method

1 Introduction

The standard random walk on the non-negative integers may be visualized by the following graph (only the first 8 states are shown):

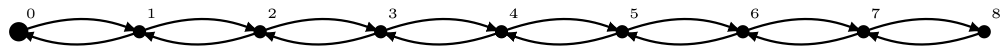

Figure 1: Standard symmetric random walk on the non-negative integers

One starts in state 0 and can go up/down one step, each with the same probability. Various questions have been studied, like the probability to end in state k. For $k = 0$, this leads to the celebrated Catalan numbers, one of the most important sequences in Discrete Mathematics. Catalan numbers appear as sequence A000108 in the Encyclopedia of Integer Sequences, Sloane (2018).

Böhm and Hornik (2010) introduced a related model: up-steps occur with probability α and down-steps occur with probability $\beta = 1-\alpha$, but after each step α and β change their roles. One motivation of this model was to study flexible chain modules in chemistry. The following graph is useful to grasp the idea. Comparing this model with the standard random walk, the only differences are the colored edges, representing the weights/probabilities. When one is in an even-indexed state (the walk has a current even height), it goes up with probability α; for the odd-indexed states it goes up with probability β.

Böhm and Hornik (2010) consider random walks on the non-negative integers and on the full set of integers as well. Alternative/additional analysis can be found in Panny and Prodinger (2016).

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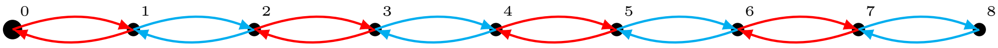

Figure 2: Red edges are labelled with the weight α, blue edges with β

Another twist of a random walk occurs in a model introduced by Knödel Knödel (1983). This idea fascinated me for 40 years. Although Knödel's paper appeared in *EIK* (Elektronische Informationsverarbeitung und Kybernetik), which is the predecessor of *JALC* (Journal of Automata, Languages and Combinatorics), it is hard to find these days. All we need to know is the following: There are bins of size 1 and small items (size $\frac{1}{3}$) and large items (size $\frac{2}{3}$) arrive with the same probability. There is an online strategy to fill up the bins. A large item can only go into an empty (new) box, but a small item can be used to complete a box that contains already a large item, thus reducing the number of incomplete content by one. There is one notable exception, namely that there is currently no box filled with a large item. Then the small item must go into a fresh box, according to the online strategy.

Here is the graph, related to the original Knödel problem:

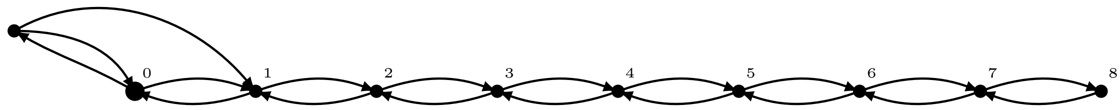

Figure 3: The original Knödel graph

States correspond to boxes filled with just one large item each. There is one exception, when a small item arrives at the origin. In this case, it cannot be used to complete a partially filled bin, and an extra state is introduced. See Prodinger (2003) and some referenced papers for analysis.

It is the purpose of this paper to combine the ideas of Knödel and Böhm-Hornik: Large items arrive with probability α and small items with probability β, but after each step the roles of α and β are changed. We do not claim that this has too much practical applications at this stage, rather the intention is to produce an interesting showcase of what is doable with modern tools of computer algebra.

The graph with two layers of states will explain the scenario readily. It is to be noted that once one goes to the special state and leaves it via another small item, the roles of even and odd change! That is why we have now a copy of each state, relative to this change; another change brings us back to the original situation. The rest of the paper is devoted to derive generating functions for walks starting at the origin and ending in a prescribed state. The kernel method Prodinger (2003) and the heavy use of computer algebra (Maple) will be essential. This method expresses unknown generating functions as a rational function. One or more factors of the denominators do *not* possess a power series expansion, which would contradict the combinatorial origin of the problems, hence these factors must cancel out. For that, division with remainder of the numerator is applied, and, knowing that there *is* no remainder, useful relations may be derived. Some people multiply the whole equation by the denominator and speak about the 'kernel', whence the name.

First, in Section 2 we start with a direct approach, which is a brute-force procedure. It leads to four equations, and eventually to biquadratic equations. Computers are capable of handling this, but the next section mostly serves as an invitation to a more sophisticated approach, using only two

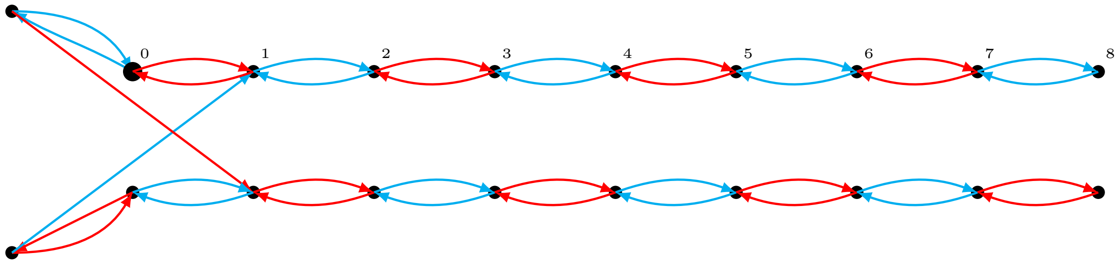

Figure 4: The Knödel-Böhm-Hornik graph

functions (not four). And, lo and behold, after a certain substitution, the ugly beast turns into a beautiful swan.

2 Brute-force Analysis

We introduce the following generating functions: $f_i = f_i(z)$ has as coefficient of z^n the probability to reach state i from the upper layer in n steps, starting from the origin (state 0). The function g_i is similar, but refers to the lower layer of states. Finally, the extra states and their generating functions are called P resp. Q. $P = P(z)$ is the generating function of all paths ending in the special state in the top layer, and $Q = Q(z)$ is the generating function of all paths ending in the special state in the bottome layer.

From the diagram (Figure 4), considering the last step made, one can see the recursions

$$\begin{aligned}
f_i &= \beta z f_{i-1} + \alpha z f_{i+1}, \ i = 2, 4, 6, \dots, \\
f_i &= \alpha z f_{i-1} + \beta z f_{i+1}, \ i = 3, 5, 7, \dots, \\
f_1 &= \alpha z f_0 + \beta z f_2 + \beta z Q = \alpha z f_0 + \beta z f_2 + \beta\alpha z^2 g_0, \\
f_0 &= 1 + \beta z P + \alpha z f_1 = 1 + \beta^2 z^2 f_0 + \alpha z f_1, \\
P &= \beta z f_0, \\
g_i &= \alpha z g_{i-1} + \beta z g_{i+1}, \ i = 2, 4, 6, \dots, \\
g_i &= \beta z g_{i-1} + \alpha z g_{i+1}, \ i = 3, 5, 7, \dots, \\
g_1 &= \beta z g_0 + \alpha z g_2 + \alpha z P = \beta z g_0 + \alpha z g_2 + \alpha\beta z^2 f_0, \\
g_0 &= \alpha z Q + \beta z g_1 = \alpha^2 z^2 g_0 + \beta z g_1, \\
Q &= \alpha z g_0.
\end{aligned} \tag{1}$$

In order to attack this system, we introduce a second variable u and consider the following four bivariate generating functions:

$$\begin{aligned}
F_e(u) &= \sum_{i\ge 0} u^{2i} f_{2i}, \quad F_o(u) = \sum_{i\ge 0} u^{2i+1} f_{2i+1}, \\
G_e(u) &= \sum_{i\ge 0} u^{2i} g_{2i}, \quad G_o(u) = \sum_{i\ge 0} u^{2i+1} g_{2i+1};
\end{aligned}$$

'e' stands for 'even', 'o' stands for odd. Summing the first recursion over all possible values of i, we find (omitting the variable u for the moment)

$$F_e - f_0 = \beta z u F_o + \frac{\alpha z}{u}(F_o - u f_1);$$

adding the recursion for f_0 (see (1)) leads to

$$F_e = \beta z u F_o + \frac{\alpha z}{u} F_o + 1 + \beta^2 z^2 f_0.$$

Similarly, for the odd indices

$$F_o - u f_1 = \alpha z u (F_e - f_0) + \frac{\beta z}{u}(F_e - f_0 - u^2 f_2)$$

and further (again using (1), as also for the following two equations)

$$F_o = \alpha z u F_e + \frac{\beta z}{u}(F_e - f_0) + u \beta \alpha z^2 g_0.$$

The same procedure is done for the even indices and the g_i's:

$$G_e - g_0 = \alpha z u G_o + \frac{\beta z}{u}(G_o - u g_1)$$

and

$$G_e = \alpha z u G_o + \frac{\beta z}{u} G_o + \alpha^2 z^2 g_0.$$

Finally, for the odd indices

$$G_o - u g_1 = \beta z u (G_e - g_0) + \frac{\alpha z}{u}(G_e - g_0 - u^2 g_2)$$

and

$$G_o = \beta z u G_e + \frac{\alpha z}{u}(G_e - g_0) + u \alpha \beta z^2 f_0.$$

For the reader's convenience we collected the four equations that we (and Maple) have to deal with:

$$\begin{aligned} F_e &= \beta z u F_o + \frac{\alpha z}{u} F_o + 1 + \beta^2 z^2 f_0, \\ F_o &= \alpha z u F_e + \frac{\beta z}{u}(F_e - f_0) + u \beta \alpha z^2 g_0, \\ G_e &= \alpha z u G_o + \frac{\beta z}{u} G_o + \alpha^2 z^2 g_0, \\ G_o &= \beta z u G_e + \frac{\alpha z}{u}(G_e - g_0) + u \alpha \beta z^2 f_0; \end{aligned}$$

we note again that $f_0 = F_e(0)$ and $g_0 = G_e(0)$; these are the two equations describing the paths returning to state 0 in the upper/lower layer. Note further that plugging in $u = 0$ just leads to tautologies of the type $f_0 = f_0$.

Maple can solve this, but the solution is implicit since it still depends on f_0 and g_0. The expressions are quite long, and they all share the same denominator D:

$$D = u^2 - z^2 u^4 \alpha - z^2 u^2 + 2u^2 \alpha z^2 + \alpha^2 z^2 u^4 - 2u^2 \alpha^2 z^2 - z^2 \alpha + z^2 \alpha^2.$$

Note here and in other places that the order in which terms appear is how it comes out from the symbolic computation. The numerators DF_e, DF_o, DG_e, DG_o are very long and not displayed here. The interested reader can find them on the website arxiv.org where a version of this material has been presented including these expressions. It is to be noted that we replaced β by $1-\alpha$ to help Maple.

The denominator D has 4 roots, considering u as the variable:

$$s_1 = \frac{\sqrt{\alpha(1-\alpha)(1-2z^2\alpha^2+2z^2\alpha-z^2-\sqrt{(1-z)(1+z)(1-z+2z\alpha)(1+z-2z\alpha)})}}{\sqrt{2}z\alpha(1-\alpha)},$$

$$s_2 = -s_1,\ s_3 = \frac{1}{s_1},\ s_4 = \frac{1}{s_2}.$$

The factors $u-s_1$ and $u-s_2$ are 'bad' in the sense of the kernel method Prodinger, 2003/04, i. e., they do not lead to a power series expansion around the origin. Consequently, the numerators of the four functions must be divisible by both factors. Applying this principle to F_e and G_e leads to two equations, from which f_0 and g_0 can be computed. Again, the expressions are long, and an auxiliary quantity W is used:

$$W = \sqrt{(1-z)(z+1)(1-z+2z\alpha)(1+z-2z\alpha)}.$$

Here are the results:

$$f_0 = \frac{\Xi_1}{4\alpha^2z^4(-1+z)(z+1)(-1+\alpha)^2(-1+z^2-3z^2\alpha+3z^2\alpha^2)},$$

$$g_0 = \frac{\Xi_2}{8z^7(-1+z)(z+1)(-1+\alpha)^4(-1+z^2-3z^2\alpha+3z^2\alpha^2)\alpha^3}.$$

The long expressions for Ξ_1 and Ξ_2 can be found in the version on arxiv.org. Plugging these results in and simplifying, we find explicit expressions for all four generating functions of interest, again with a common denominator M: $M = (-1+z)(z+1)(-1+\alpha)^3(-1+z^2-3z^2\alpha+3z^2\alpha^2)(u^2-z^2u^4\alpha-z^2u^2+2u^2\alpha z^2+\alpha^2z^2u^4-2u^2\alpha^2z^2-z^2\alpha+z^2\alpha^2)$. With this, the numerators Mf_o, Mf_e, Mg_o, Mg_e become fully explicit; again, they are quite long and not displayed here.

Of course, the expressions are not appealing, but that is what the brute-force approach produces. The full information of all the states in the graph is packed into the expressions. We will see later that distinguishing between even and odd makes the problem simpler.

We can derive as many corollaries from this as we want, of course with Maple:

$$f_0 = [u^0]F_e = 1+(2\alpha^2+1-2\alpha)z^2+(5\alpha^4-10\alpha^3+9\alpha^2-4\alpha+1)z^4+\cdots,$$
$$f_1 = [u^1]F_o = \alpha z+(3\alpha^2-4\alpha+2)\alpha z^3+(8\alpha^4-19\alpha^3+20\alpha^2-11\alpha+3)\alpha z^5+\cdots,$$
$$f_2 = [u^2]F_e = \alpha(1-\alpha)z^2+2(1-\alpha)(2\alpha^2+1-2\alpha)\alpha z^4+\cdots,$$
$$f_3 = [u^3]F_o = (1-\alpha)\alpha^2z^3+(1-\alpha)(5\alpha^2-6\alpha+3)\alpha^2z^5+\cdots,$$

and similarly

$$g_0 = [u^0]G_e = \alpha(1-\alpha)^2z^3+(5\alpha^2-4\alpha+2)(1-\alpha)^2\alpha z^5+\cdots,$$
$$g_1 = [u^1]G_o = \alpha(1-\alpha)z^2+2(1-\alpha)(2\alpha^2+1-2\alpha)\alpha z^4,$$
$$g_2 = [u^2]G_e = (1-\alpha)\alpha^2z^3+(1-\alpha)(5\alpha^2-6\alpha+3)\alpha^2z^5+\cdots,$$
$$g_3 = [u^3]G_o = \alpha^2(1-\alpha)^2z^4+3\alpha^2(2\alpha^2+1-2\alpha)(1-\alpha)^2z^6+\cdots.$$

These are the power series expansions of the first few terms in f_i and g_i for $i=0,1,2,3$.

3 A more sophisticated approach

The imbalance of α versus β is leveled out after 2 (or an even number of) steps. Thus, as in Panny and Prodinger (2016), we consider the system after an even number of steps. In the following graph, a directed arrow stands for 2 steps (a double-step). Note that the system is still working without look-ahead (following Knödel's idea about the online strategy), writing s for the small item of size $\frac{1}{3}$ and l for the large item of size $\frac{2}{3}$, the sequences (double-steps) sl resp. ls lead to different states when being in the special state named Q. Note that we have only 'half' (with a grain of salt) of the states compared to the brute-force approach.

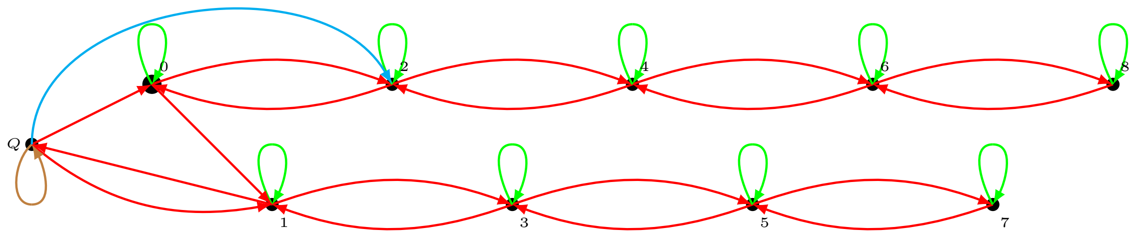

Figure 5: Two steps. Red with probability $\alpha\beta$, green with probability $1-2\alpha\beta=\alpha^2+\beta^2$, blue with probability β^2, brown with probability α^2.

The graph is now simpler than before. We introduce generating functions f_N for the upper layer of states, and g_N for the lower layer of states. The meaning of these generating functions is now different from the previous section, but it is apparent how they are related. Here are the recursions:

$$\begin{aligned}
f_N &= z\alpha\beta f_{N-1} + z\alpha\beta f_{N+1} + z(\alpha^2+\beta^2) f_N, \quad N \ge 2,\\
f_1 &= z\alpha\beta f_0 + z\alpha\beta f_2 + z(\alpha^2+\beta^2) f_1 + z\beta^2 f_Q,\\
f_0 &= 1 + z\alpha\beta f_1 + z(\alpha^2+\beta^2) f_0 + z\alpha\beta f_Q,\\
f_Q &= z\alpha\beta g_0 + z\alpha^2 f_Q = \frac{z\alpha\beta g_0}{1-z\alpha^2},\\
g_N &= z\alpha\beta g_{N-1} + z\alpha\beta g_{N+1} + z(\alpha^2+\beta^2) g_N, \quad N \ge 1,\\
g_0 &= z\alpha\beta f_0 + z\alpha\beta g_1 + z\alpha\beta f_Q + z(\alpha^2+\beta^2) g_0.
\end{aligned}$$

Introducing only *two* bivariate generating functions

$$F(u) = \sum_{N\ge 0} u^N f_N \quad \text{and} \quad G(u) = \sum_{N\ge 0} u^N g_N,$$

we find by summing the recursions over all possible indices

$$\begin{aligned}
F(u) = \sum_{N\ge 0} u^N f_N &= z\alpha\beta \sum_{N\ge 2} u^N f_{N-1} + z\alpha\beta \sum_{N\ge 2} u^N f_{N+1} + z(\alpha^2+\beta^2) \sum_{N\ge 2} u^N f_N\\
&\quad + u\Big(z\alpha\beta f_0 + z\alpha\beta f_2 + z(\alpha^2+\beta^2) f_1 + z\beta^2 \frac{z\alpha\beta g_0}{1-z\alpha^2}\Big)
\end{aligned}$$

$$\begin{aligned}&+1+z\alpha\beta f_1+z(\alpha^2+\beta^2)f_0+z\alpha\beta\frac{z\alpha\beta g_0}{1-z\alpha^2}\\&=z\alpha\beta uF(u)+\frac{z\alpha\beta}{u}(F(u)-f_0)+z(\alpha^2+\beta^2)F(u)\\&+uz^2\beta^3\alpha\frac{g_0}{1-z\alpha^2}+1+z^2\alpha^2\beta^2\frac{g_0}{1-z\alpha^2}\end{aligned}$$

and

$$\begin{aligned}G(u)=\sum_{N\geq 0}u^Ng_N&=z\alpha\beta\sum_{N\geq 1}u^Ng_{N-1}+z\alpha\beta\sum_{N\geq 1}u^Ng_{N+1}+z(\alpha^2+\beta^2)\sum_{N\geq 1}u^Ng_N\\&+z\alpha\beta f_0+z\alpha\beta g_1+z\alpha\beta\frac{z\alpha\beta g_0}{1-z\alpha^2}+z(\alpha^2+\beta^2)g_0\\&=z\alpha\beta uG(u)+\frac{z\alpha\beta}{u}(G(u)-g_0)+z(\alpha^2+\beta^2)G(u)\\&+z\alpha\beta f_0+z^2\alpha^2\beta^2\frac{g_0}{1-z\alpha^2}.\end{aligned}$$

Solving the system leads to

$$\begin{aligned}F(u)&=\frac{-uz\alpha^2+z^2\alpha^2\beta^2g_0u-z\alpha\beta f_0+z^2\alpha^3\beta f_0+u+u^2z^2\beta^3\alpha g_0}{u-2uz\alpha^2-z\alpha\beta u^2+z^2\alpha^3\beta u^2-z\alpha\beta+z^2\alpha^3\beta+z^2u\alpha^4-zu\beta^2+z^2u\beta^2\alpha^2},\\G(u)&=-\frac{z\alpha\beta(-z\alpha\beta g_0u+g_0-g_0z\alpha^2+z\alpha^2f_0u-f_0u)}{u-2uz\alpha^2-z\alpha\beta u^2+z^2\alpha^3\beta u^2-z\alpha\beta+z^2\alpha^3\beta+z^2u\alpha^4-zu\beta^2+z^2u\beta^2\alpha^2}.\end{aligned}$$

These answers are implicit, since they contain $f_0=F(0)$ and $g_0=G(0)$. Plugging in $u=0$ at that stage would just lead to tautologies like $f_0=f_0$ and $g_0=g_0$. To remedy the situation and make the expression explicit, the kernel method is used once again. As before, after cancelling the 'bad factor ' from numerator resp. denominator (in the spirit of the kernel method) in both instances, one *can* solve! The denominators factor as

$$z\alpha\beta(-1+z\alpha^2)(u-r_1)(u-r_2)$$

with

$$r_2=\frac{1-z\alpha^2-z\beta^2-\sqrt{z^2\alpha^4-2z^2\beta^2\alpha^2-2z\alpha^2+z^2\beta^4-2z\beta^2+1}}{2z\alpha\beta}$$

and $r_1=\frac{1}{r_2}$. Notice that our more sophisticated treatment does not make it necessary to replace β by $1-\alpha$.

The factor $(u-r_2)$ (the 'bad' factor) must cancel from numerator and denominator. Maple can perform the division with remainder with little effort. All we have to do then is to ignore the remainder since we know a priori that is must be 0. The result is now

$$F(u)=\frac{r_2z^2\beta^3\alpha g_0-z\alpha^2+z^2\alpha^2\beta^2g_0+1+uz^2\beta^3\alpha g_0}{z\alpha\beta(-1+z\alpha^2)(u-r_1)}$$

and

$$G(u)=\frac{-(-z\alpha\beta g_0+z\alpha^2f_0-f_0)}{(-1+z\alpha^2)(u-r_1)}.$$

Plugging in $u = 0$ leads now to manageable equations,

$$f_0 = \frac{r_2 z^2 \beta^3 \alpha g_0 - z\alpha^2 + z^2\alpha^2\beta^2 g_0 + 1}{z\alpha\beta(-1 + z\alpha^2)(-r_1)},$$
$$g_0 = \frac{(-z\alpha\beta g_0 + z\alpha^2 f_0 - f_0)}{(-1 + z\alpha^2)r_1}.$$

From these, we can compute f_0 and g_0 easily, but do not print it, since it is not too attractive at the moment (in a moment, it will become very beautiful).

It is easy to see by reading off the coefficient of u^j that

$$[u^j]G(u) = \frac{(-z\alpha\beta g_0 + z\alpha^2 f_0 - f_0)}{(-1 + z\alpha^2)} r_2^{j+1}$$

and

$$[u^j]F(u) = -r_2^{j+1} \frac{r_2 z^2 \beta^3 \alpha g_0 - z\alpha^2 + z^2\alpha^2\beta^2 g_0 + 1}{z\alpha\beta(-1 + z\alpha^2)} - r_2^j \frac{z\beta^2 g_0}{(-1 + z\alpha^2)}.$$

This is easy since the variable u appears only once in the respective denominators. To say it again, f_0 and g_0 are known functions at this stage (just not printed).

Note that $[z^m u^j]F(u)$ is the probability to reach state $2j$ in m (double-)steps, and $[z^m u^j]G(u)$ is the probability to reach state $2j + 1$ in m (double-)steps.

More attractive formulæ thanks to a substitution

Using the substitution

$$z = \frac{v}{\alpha\beta + (\alpha^2 + \beta^2)v + \alpha\beta v^2} = \frac{v}{(\alpha + v\beta)(\beta + v\alpha)},$$

(inspired by our old paper Panny and Prodinger, 2016) all the expressions become nicer.[1] For instance, $r_2 = v$ and

$$f_0 = \frac{(v\alpha + \beta)(\alpha + v\beta)}{\alpha\beta(1 - v)(v^2 + v + 1)},$$
$$g_0 = \frac{v(\alpha + \alpha v^2 + v\beta)(v\alpha + \beta)}{\alpha\beta(1 - v)(v^2 + v + 1)}.$$

The equality $(1 - v)(v^2 + v + 1) = 1 - v^3$ might be useful as well. Even the full bivariate generating functions look now very nice:

$$F = F(z, u) = \frac{(uv^3\beta + \alpha + v\beta)(v\alpha + \beta)}{\beta\alpha(1 - uv)(1 - v)(v^2 + v + 1)},$$
$$G = G(z, u) = \frac{v(\alpha + \alpha v^2 + v\beta)(v\alpha + \beta)}{\beta\alpha(1 - uv)(1 - v)(v^2 + v + 1)}.$$

[1] Compare this with the *Joukowsky transform*, https://en.wikipedia.org/wiki/Joukowsky_transform.

Consequently, reading off coefficient of powers of u,

$$[u^j]F = \frac{v^j(\alpha+v\beta)(v\alpha+\beta)}{\beta\alpha(1-v)(v^2+v+1)} + \frac{v^{j+1}}{\alpha(1-v)(v^2+v+1)}$$

and

$$[u^j]G = \frac{v^{j+1}(\alpha+\alpha v^2+v\beta)(v\alpha+\beta)}{\beta\alpha(1-v)(v^2+v+1)}.$$

Finally we answer the question how to read off coefficients of powers of z when the function is given in terms of v. We describe in this way how to switch back from the v-notation to the z-notation.

For that, we employ Cauchy's integral formula in the following computation,

$$\begin{aligned}[z^N]H(z(v)) &= \frac{1}{2\pi i}\oint \frac{dz}{z^{N+1}}H(z(v)) \\ &= \frac{1}{2\pi i}\oint \frac{dv}{v^{N+1}}\frac{\alpha\beta(1-v^2)}{(\alpha+\beta v)(\beta+\alpha v)}(\alpha+\beta v)^{N+1}(\beta+\alpha v)^{N+1}H(v) \\ &= [v^N]\alpha\beta(1-v^2)(\alpha+\beta v)^N(\beta+\alpha v)^N H(v).\end{aligned}$$

For interest, we state that

$$v = \frac{1-z+2z\alpha\beta-\sqrt{(1-z)(1-z+4z\alpha\beta)}}{2z\alpha\beta}.$$

Walks with an odd number of steps

For that, we do not need to do new calculations, by considering the last step separately. We refer to the original Figure 4. It is immediate to see that

$$\begin{aligned}&\mathbb{P}\{\text{reach top level state } 2j+1 \text{ in } 2m+1 \text{ steps}\} \\ &\qquad = \alpha\mathbb{P}\{\text{reach top level state } 2j \text{ in } 2m \text{ steps}\} \\ &\qquad + \beta\mathbb{P}\{\text{reach top level state } 2j+2 \text{ in } 2m \text{ steps}\}\end{aligned}$$

and

$$\begin{aligned}&\mathbb{P}\{\text{reach bottom level state } 2j \text{ in } 2m+1 \text{ steps}\} \\ &\qquad = \alpha\mathbb{P}\{\text{reach bottom level state } 2j-1 \text{ in } 2m \text{ steps}\} \\ &\qquad + \beta\mathbb{P}\{\text{reach bottom level state } 2j+1 \text{ in } 2m \text{ steps}\};\end{aligned}$$

the exceptional cases near the beginning are easy to figure out directly. Of course these considerations are simple observations using the previously obtained results.

4 Asymptotics

Although this paper concentrates on *explicit enumerations*, some asymptotic considerations might be of interest. One natural concept would the height of a Knödel walk, i. e., the state with the highest index that is reached during the walk. For simpler walks, this has been worked out in Prodinger (1992), compare Panny and Prodinger (2016). However, that would be a completely different approach, and we have chosen the kernel method as the unifying method of choice in this paper.

We can, however, offer something appealing here, namely we compute the average index of the state where the walks ends (in the sophisticated version). So we compute

$$\text{EXPECTED-END} = \sum_{k\ge 0}(2k)f_k + \sum_{k\ge 0}(2k+1)g_k.$$

This is best computed using the bivariate generating functions:

$$\begin{aligned}\text{EXPECTED-END} &= \frac{\partial}{\partial u}\Big(F(u^2,z)+uG(u^2,z)\Big)\Big|_{u=1}\\ &= \frac{v\,(v\alpha+\beta)\left(3v^2\beta+3\alpha+3v\beta+v\alpha+\alpha v^2+\alpha v^3\right)}{\alpha\beta\,(1-v)^3\,(1+v+v^2)}.\end{aligned}$$

To find asymptotics, we transfer back from v to z. In order to avoid ungainly expressions that the reader can generate himself/herself with a computer, we demonstrate the procedure for the standard case $\alpha=\beta=\frac{1}{2}$. Then

$$v = \frac{-z+2-2\sqrt{1-z}}{z} \sim \frac{z}{4}+\frac{z^2}{8}+\cdots$$

and

$$\text{EXPECTED-END} = \frac{z-1+(1+z)\sqrt{1-z}}{2(1-z)^2} \sim \frac{1}{(1-z)^{3/2}}.$$

The coefficient of z^n in this expression is $\binom{-3/2}{n}(-1)^n \sim 2\sqrt{\frac{n}{\pi}}$, which is the answer to the question about the average index where the walk stops. In the general instance, the singularity of interest is $z\sim 1$, which is equivalent to $v\sim 1$, and

$$1-v \sim \frac{1}{\sqrt{\alpha\beta}}\sqrt{1-z}.$$

But

$$\text{EXPECTED-END} \sim \frac{2}{\alpha\beta}\frac{1}{(1-v)^3} \sim 2\sqrt{\alpha\beta}\frac{1}{(1-z)^{3/2}},$$

whence the result in the general case is $\sim 4\sqrt{\alpha\beta}\sqrt{\frac{n}{\pi}}$.

The method of local expansions around the dominant singularity (here $z=1$) and then translating into the behaviour of the coefficients is called *singularity analysis of generating functions*. Standard references are Flajolet and Sedgewick (2009) and Flajolet and Odlyzko (1990).

One could treat this particular problem as well with more elementary methods, but since the generating functions were already at hand, we just used them. Again, this example just highlights the type of asymptotic investigations that one could make. More complicated questions would not fit in here.

5 Conclusion

We want to emphasize the following points:

- A brute-force approach is possible, but leads to equations of order 4 and explicit but very ungainly expressions.
- Looking at the system after an even number of steps is a clever idea, since the imbalance of α versus β is leveled out. The equations are only quadratic.
- Introducing an auxiliary variable, all the generating functions become rational (in the variable). Consequently reading off coefficients is not difficult.
- To go from an even number of steps to an odd number of steps is not difficult, when considering the last step separately and use previous results.
- Once the generating functions of interest are known explicitly, several corollaries of an asymptotic nature can be derived from them.

One might wonder which lattice path problems *are* amenable to the kernel method. A general answer is difficult. The following texts contain a variety of examples: Banderier and Flajolet (2002), Prodinger (2003), Prodinger (2023).

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